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A MODEL FOR AERIAL SURVEILLANCE OF MOVING OBJECTS
WHEN ERRORS OF OBSERVATION ARE MULTI-VARIATE NORMAL

by

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A MODEL FOR AERIAL SURVEILLANCE OF MOVING OBJECTS
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This paper presents a general theorem on the invariant behavior of
a certain function of a matrix. It then shows the importance of this
result principally by using it to derive properties of certain maximum
likelihood estimates which arise when considering problems such as the
location of a moving object being surveyed from a moving observatory
when all data on location are subject to stochastic error. This problem
is important in tracking objects either from an observatory satellite or
from a transport plane bearing ground seeking radar. Some applications to
this situation are made.

1. INTRODUCTION

This paper stems, in part, from two earlier proprietary reports, viz. Saunders (1965) and Saunders and Johnson (1964), dealing with statistical problems arising from the estimation of the position, heading and velocity of a moving object using data which are subject to statistical error. These data were presumed to have been obtained from observations made during one overflight when the exact location of the observatory platform is not precisely known with respect to the ground. The solution of this problem was originally intended to assist in the determination of both the travel and location of ships when using data obtained from an observatory satellite. Recently the same mathematical problems have arisen with the introduction of ground mapping radar which is being born by airplanes and used in the observation and prediction of position of moving land vehicles.

In the first section a theorem on the behavior of a certain function of a matrix is stated and proved. In the following sections a simple model with normal errors for the moving traget from a moving observatory is given and the maximum likelihood estimates of target position are obtained. The theorem is then utilized to yield certain invariance properties of these estimates.

2. THE GENERAL THEOREM

We now state and prove a general result on the behavior of a particular function of positive definite matrices.

<u>Theorem 1</u>: Let c be a given real number, Σ_0, Σ_1, Σ_2 positive definite (symmetric) matrices and the matrix function $F(t)$, defined for any real t by the expression

$$F(t) = (I, (t+c)I) \begin{pmatrix} \Sigma_0 & \Sigma_1 + t\Sigma_0 \\ \Sigma_1 + t\Sigma_0 & \Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 \end{pmatrix}^{-1} \begin{pmatrix} I \\ (t+c)I \end{pmatrix},$$

then $F(t) \equiv F(0)$.

<u>Proof.</u> The result will hold if $f(t) = uF(t)u'$ is independent of t for all vectors u of appropriate dimension. Note

$$f(t) = (u, (t+c)u) \begin{pmatrix} \Sigma_0 & \Sigma_1 + t\Sigma_0 \\ \Sigma_1 + t\Sigma_0 & \Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 \end{pmatrix}^{-1} \begin{pmatrix} u' \\ (t+c)u' \end{pmatrix}.$$

Further note that if A is non-singular, then

$$|A + w'w| = |A||I + A^{-1}w'w| = |A|(1 + wA^{-1}w'),$$

so that

$$wA^{-1}w' = \frac{|A + w'w|}{|A|} - 1.$$

Consequently, $f(t)$ independent of t is equivalent to

$$(2.1) \qquad \frac{\begin{vmatrix} \Sigma_0 + u'u & \Sigma_1 + t\Sigma_0 + (t+c)u'u \\ \Sigma_1 + t\Sigma_0 + (t+c)u'u & \Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 + (t+c)^2u'u \end{vmatrix}}{\begin{vmatrix} \Sigma_0 & \Sigma_1 + t\Sigma_0 \\ \Sigma_1 + t\Sigma_0 & \Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 \end{vmatrix}}$$

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independent of t. If the numerator is independent of t for all u then the denominator will also be independent of t by taking u = 0. Using the fact that if A is non-singular

$$\begin{vmatrix} A & B \\ B' & D \end{vmatrix} = |A| |D - B'A^{-1}B| \ ,$$

we obtain for the numerator of (2.1);

$$|\Sigma_0 + u'u| \cdot |\Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 + (t+c)^2 u'u -$$

$$[\Sigma_1 + cu'u + t(\Sigma_0 + u'u)](\Sigma_0 + u'u)^{-1}(\Sigma_1 + t\Sigma_0 + (t+c)u'u)| \ .$$

The first term is independent of t. The matrix in the second term is

$$\Sigma_2 + 2t\Sigma_1 + t^2\Sigma_0 + (t+c)^2 u'u - (\Sigma_1 + cu'u)(\Sigma_0 + u'u)^{-1}(\Sigma_1 + cu'u)$$

$$- 2t(\Sigma_1 + cu'u) - t^2(\Sigma_0 + u'u) \ ,$$

which upon expansion and simplification reduces to

$$\Sigma_2 + c^2 u'u - (\Sigma_1 + cu'u)(\Sigma_0 + u'u)^{-1}(\Sigma_1 + cu'u) \ .$$

This expression is independent of t, which completes the proof. ‖

3. THE MODEL AND ITS ASSUMPTIONS

We shall speak of the object (or target) and the observatory, including thereby all applications, with the understanding that both points are moving

with respect to a given coordinate axis. Our first task is to derive the appropriate density of the observations of target positions relative to a fixed coordinate axis, as determined from the observatory.

We now specify precisely the assumptions on which our analysis is based. These are:

1° The observatory moves at constant height linearly above the plane with a known constant velocity.

2° The object moves linearly in the plane at a constant but unknown velocity.

3° The estimated position coordinates of the observatory over the plane, as determined from the ground at a given time, are bivariate normal random variables with known covariance.

4° The estimated position coordinates of the object in the plane, as determined relative to the true position of the observatory, are bivariate normal random variables and successive observations of such relative positions are independent.

5° Time between successive observations can be measured with sufficient accuracy so that errors of position due to time inaccuracy are negligible.

6° The parameters of the covariance matrix of the observations of object position relative to the true observatory position can be determined from bearing angle and range data.

On a single overflight, an observatory may make several observations of the position of an object and our first problem is to determine the joint distribution of the observations of the target position, using the fact that the target position relative to the observatory is subject to

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observational error, as is the estimate of the observatory positions relative to the ground. Thus, at a given time t_i we assume the observatory is at some position, say ρ_i, and the target at some position μ_i both in a plane located with respect to a given coordinate system. However, the position of the object as observed by radar from the observatory is subject to chance error and hence, the radar estimate of the object position from the observatory position is a random variable, say X_i. Now a radar measurement from the ground at time t_0 of the observatory position ρ_0 on the given coordinate system is also a random variable, call it Y.

From assumptions 1° and 2° we have that the observatory follows a linear path in the plane, say $\rho_t = \iota + \varepsilon t$, as does the object, say, $\mu_t = \alpha + \beta t$. (Greek letters denote points in the plane.)

Without loss of generality, we can select our coordinates so that the first coordinate of ε is in the direction of observatory travel and hence, the second coordinate of ε is zero. Moreover, by assumption 1°, the first coordinate of ε is known.

Again, following the general mathematical assumptions 3° and 4°, we have that X_i, for $i = 1,\ldots,n$, and Y are bivariate normal. More specifically, X_i has mean vector $\mu_i - \rho_i$ and known covariance matrix C_i^{-1}, whereas Y has mean vector ρ_0 and covariance matrix D^{-1}. That is, for $i = 1,\ldots,n$

$$X_i \sim \eta(\mu_i - \rho_i, C_i^{-1}) \quad \text{and} \quad Y \sim \eta(\rho_0, D^{-1}) .$$

The data obtained from the observatory yield the observations $Z_i = X_i + Y$ for $i = 1,\ldots,n$ and in this section we seek to derive the joint density, f, of $Z = (Z_1,\ldots,Z_n)$. Since $Z_i = X_i + Y$, it follows

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that $\text{Var}(Z_i) = C_i^{-1} + D^{-1}$ and $\text{Cov}(Z_i, Z_j) = D^{-1}$ for $i \neq j$,
$i, j = 1, \ldots, n.$ Hence

$$\Sigma = \text{Cov}(Z) = (\delta_{ij} C_i^{-1} + D^{-1}) ,$$

where δ_{ij} is the Kronecker delta. Setting

$$\Sigma^{-1} = \Lambda = (\Lambda_{ij})$$

it follows that

$$(3.1) \qquad \Lambda_{ij} = \delta_{ij} C_i - C_i \left(D + \sum_1^n c_\ell \right)^{-1} C_j .$$

This inversion is a consequence of the

Lemma: If $Q_1, \ldots, Q_k$ are non-singular symmetric matrices of the same
dimension and

$$H = \text{diag}(Q_1, \ldots, Q_k) + \begin{pmatrix} M_1 \\ \vdots \\ M_k \end{pmatrix} (I, \ldots, I) ,$$

then

$$H^{-1} = \text{diag}(Q_1^{-1}, \ldots, Q_k^{-1}) - \begin{pmatrix} Q_1^{-1} M_1 \\ \vdots \\ Q_k^{-1} M_k \end{pmatrix} \left(I + \sum_{i=1}^k Q_i^{-1} M_i \right)^{-1} (Q_1^{-1}, \ldots, Q_k^{-1}) .$$

See Householder (1964), p. 124.

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Specifically, the choice of $Q_i = C_i^{-1}$, $M_i = D^{-1}$ yields the result claimed.

Noting that $\Lambda'_{ij} = \Lambda_{ji}$, it follows that Λ is itself symmetric. Except for elementary details we have obtained the basic

Theorem 2: The distribution of the observations of target position $Z = (Z_1, \ldots, Z_n)$, obtained from the observatory is

$$Z \sim \mathfrak{n}(\nu, \Sigma) \, ,$$

with density function

$$(3.2) \qquad f(\mathbf{z}) = \frac{|\Lambda|^{1/2}}{(2\pi)^n} \exp[-\frac{1}{2}(z-\nu)\Lambda(z-\nu)'] \, ,$$

where $z \in R^{2n}$, $\nu = (\nu_1, \ldots, \nu_n) = (\mu_1 - \rho_1 - \rho_0, \ldots, \mu_n - \rho_n - \rho_0)$, $\Lambda = \Sigma^{-1}$ is defined by (3.1), and

$$(3.2) \qquad |\Lambda| = \frac{|D| \prod_1^n |C_i|}{|D + \sum_1^n C_i|} \, .$$

Theorem 2 tells us that a single observation of observatory position from a bivariate normal distribution relative to the ground, combined with n observations of the object position, which are normally distributed relative to the observatory position yields a joint normal distribution of observed target positions relative to the ground. Our next problem is to find a best estimate of the target course using this normal distribution of error and the assumption that the target is moving linearly at a constant velocity.

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4. THE MAXIMUM LIKELIHOOD ESTIMATES AND THEIR DISTRIBUTION

Thus, we want to find the maximum likelihood estimate (MLE), $\hat{\mu}_t$, of future target positions μ_t, as a function of t. By properties of the MLE,

$$\hat{\mu}_t = \hat{\alpha} + \hat{\beta}t \ ,$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the MLE's of α and β. To complete the picture we need to know the distribution of $(\hat{\alpha}, \hat{\beta})$.

Recall that $\nu_i = \mu_i - \rho_i + \rho_0 = (\alpha + \varepsilon t_0) + (\beta - \varepsilon)t_i$,   for $i = 1, \ldots, n$. If we define $\gamma = \alpha + \varepsilon t_0$, $\kappa = \beta - \varepsilon$ and obtain the MLE's of γ and κ, then we can easily obtain the MLE's of α, β.

The likelihood function is, with $\zeta_i = z_i - \nu_i$ for $i = 1, \ldots, n$,

$$L = \log|E| - n \log(2\pi) - \sum_{i,j} \zeta_i \Lambda_{ij} \zeta_j' \ .$$

Introduce the notation for the coordinates of

$$\gamma = (\gamma_1, \gamma_2) \ , \quad \kappa = (\kappa_1, \kappa_2) \ ,$$

then writing $\delta_k = (\delta_{1k}, \delta_{2k})$ for $k = 1, 2$ where δ_{ij} is the Kronecker delta we obtain, by taking partial derivatives, the two equations for $k = 1, 2$,

$$(4.1) \qquad \frac{\partial L}{\partial \gamma_k} = \frac{1}{2} \sum_{i,j} (\zeta_i \Lambda_{ij} \delta_k' + \delta_k \Lambda_{ij} \zeta_j') \ ,$$

$$(4.2) \qquad \frac{\partial L}{\partial \kappa_k} = \frac{1}{2} \sum_{i,j} (t_j \zeta_i \Lambda_{ij} \delta_k' + t_i \delta_k \Lambda_{ij} \zeta_j') \ .$$

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Realizing the second term in each expression is a scalar, and thus equal
to its own transpose, the two equations in (4.1) and the two equations
in (4.2) can be written matrix form, upon equating to zero, as

$$(4.3) \qquad \sum_{i,j} (\zeta_i + \zeta_j) \Lambda_{ij} = (0,0) \; ,$$

$$(4.4) \qquad \sum_{i,j} (t_j \zeta_i + t_i \zeta_j) \Lambda_{ij} = (0,0) \; .$$

Note that

$$\zeta_i + \zeta_j = z_i + z_j - 2\gamma - \kappa(t_i + t_j) \; ,$$

$$t_j \zeta_i + t_i \zeta_j = t_j z_i + t_i z_j - \gamma(t_i + t_j) - 2\kappa t_i t_j \; .$$

By substitution, we obtain from (4.3) and (4.4) the equations

$$(4.5) \qquad \sum_{i,j} (z_i + z_j) \Lambda_{ij} = 2\gamma \sum_{i,j} \Lambda_{ij} + \kappa \sum_{i,j} (t_i + t_j) \Lambda_{ij} \; ,$$

$$(4.6) \qquad \sum_{i,j} (t_j z_i + t_i z_j) \Lambda_{ij} = \gamma \sum_{i,j} (t_i + t_j) \Lambda_{ij} + 2\kappa \sum_{i,j} t_i t_j \Lambda_{ij} \; .$$

Writing these in matrix notation, we have

$$(4.7) \qquad (\gamma, \kappa) \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} = (\psi_1, \psi_2) \; ,$$

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where

$$\Sigma_{11} = \sum_{i,j} \Lambda_{ij} \ , \ \Sigma_{12} = \Sigma_{21} = \sum_{i,j} \frac{t_i + t_j}{2} \Lambda_{ij} \ , \ \Sigma_{22} = \sum_{i,j} t_i t_j \Lambda_{ij} \ .$$

Define the matrix $F = (F_{ij})$ by $F_{ij} = \frac{1}{2}(\Lambda_{ij} + \Lambda_{ji})$ for $i,j = 1,\ldots,n$, so that $F = \frac{\Lambda + \Lambda'}{2}$ with the obvious definition of the partitioned matrix $\Lambda' = (\Lambda'_{ij})$ where $\Lambda'_{ij} = \Lambda_{ji}$. By rearranging some summations,

$$(4.8) \qquad \Sigma_{11} = \sum_{i,j} \Lambda_{ij} \ , \ \Sigma_{12} = \Sigma_{21} = \sum_{i,j} t_i F_{ij}, \ \Sigma_{22} = \sum_{i,j} t_i t_j F_{ij} \ ,$$

$$\psi_1 = \sum_{i,j} z_i F_{ij} \qquad \psi_2 = \sum_{i,j} z_i t_j F_{ij} \ .$$

If we denote

$$S^{-1} = (\Sigma_{ij}) \ , \quad S = (S_{ij}) \ , \quad i,j = 1,2$$

then (4.7) can be written as $(\hat{\gamma}, \hat{\kappa}) = (\psi_1, \psi_2)S$, with

$$(4.9) \qquad \hat{\gamma} = \psi_1 S_{11} + \psi_2 S_{21} \ , \quad \hat{\kappa} = \psi_1 S_{12} + \psi_2 S_{22} \ ,$$

where $S_{21} = S'_{12}$. We now write from the definition of ψ_i

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$$(4.10) \qquad (\psi_1, \psi_2) = (Z_1, \ldots, Z_n) \begin{pmatrix} \sum_j F_{1j} & \sum_j t_j F_{1j} \\ \vdots & \vdots \\ \sum_j F_{nj} & \sum_j t_j F_{nj} \end{pmatrix}$$

or in matrix notation, with the obvious definition,

$$\psi = ZU .$$

By Theorem 2, Z is $\eta(\nu, \Lambda^{-1})$ and ψ is $\eta(\nu U, U'\Lambda^{-1}U)$. Since $(\hat{\gamma}, \hat{\kappa}) = \psi S$, $(\hat{\gamma}, \hat{\kappa})$ is $\eta(\nu US, Q)$, where $Q = S'U'\Lambda^{-1}US$. We now prove that

$$(4.11) \qquad \qquad \nu US = (\gamma, \kappa) .$$

To do this we make use of the identities which follow from $\Sigma^{-1}S = I$. The first component of the vector νUS is

$$\sum_{i,j} (\gamma + \kappa t_i) F_{ij} S_{11} + \sum_{i,j} (\gamma + \kappa t_i) t_j F_{ij} S_{21}$$

$$= \gamma(\Sigma_{11}S_{11} + \Sigma_{12}S_{21}) + \kappa(\Sigma_{21}S_{11} + \Sigma_{22}S_{21}) = \gamma .$$

The second component of the vector νUS is

$$\sum_{i,j} (\gamma + \kappa t_i) F_{ij} S_{12} + \sum_{i,j} (\gamma + \kappa t_i) t_j F_{ij} S_{22}$$

$$= \gamma(\Sigma_{11}S_{12} + \Sigma_{12}S_{22}) + \kappa(\Sigma_{21}S_{12} + \Sigma_{22}S_{22}) = \kappa ,$$

which completes the proof of (4.11). $\|$

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We have just obtained

Theorem 3: The MLE, $(\hat{\alpha},\hat{\beta})$, of (α,β) has components given by

$$\hat{\alpha} = \hat{\gamma} - \varepsilon t_0 \; , \quad \hat{\beta} = \hat{\kappa} + \varepsilon \; ,$$

where $(\hat{\gamma},\hat{\kappa})$ are defined by (4.9). Furthermore $(\hat{\alpha},\hat{\beta}) \sim n((\alpha,\beta),Q)$ with

$$Q = S'U'\Lambda^{-1}US = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}.$$

From this we obtain

Corollary 1: If, from conditions of symmetry, the additional assumption that

(4.12)
$$\Lambda'\Lambda^{-1}\Lambda' = \Lambda$$

holds, then

(4.13)
$$Q = S'U'\Lambda^{-1}US = S \; .$$

Proof. From (4.10)

$$U = \begin{pmatrix} F_{11} & \cdots & F_{1n} \\ & \vdots & \\ F_{n1} & \cdots & F_{nn} \end{pmatrix} \begin{pmatrix} I & & t_1 I \\ & \vdots & \\ I & & t_n I \end{pmatrix}$$

Since $F' = F$, it follows that $F\Lambda^{-1}F = F$, and hence

$$U'\Lambda^{-1}U = \sum_{i,j} \begin{pmatrix} F_{ij} & t_j F_{ij} \\ t_i F_{ij} & t_i t_j F_{ij} \end{pmatrix} = \begin{pmatrix} \sum_{i,j} F_{ij} & \sum_{i,j} t_j F_{ij} \\ \sum_{i,j} t_i F_{ij} & \sum_{i,j} t_i t_j F_{ij} \end{pmatrix} = S^{-1} \; .$$

The assertion follows since $S' = S$. $\|$

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Corollary 2: The MLE $\hat{\mu}_t$ of the true target position μ_t at time t is

$$\hat{\mu}_t = \hat{\alpha} + \hat{\beta}t \sim \mathcal{n}(\mu_t, B_t) \, ,$$

where

(4.14) $$B_t = Q_{11} + t(Q_{12} + Q_{21}) + t^2 Q_{22} \, .$$

Applying Theorem 1, we have obtained a result of great practical importance, namely,

Theorem 4: The estimate $\hat{\mu}_t$ and its covariance matrix B_t are invariant under location and scale change in time.

5. SOME SPECIALIZATIONS OF PRACTICAL IMPORTANCE

Assume that a given time t_i the observatory is at point ρ_i and traveling in a straight line at a known constant velocity. It observes, at a true bearing angle θ_i and range R_i, the object which is located at position μ_i obtaining the random variable $X_i \sim \mathcal{n}(\mu_i - \rho_i, C_i^{-1})$. See Figure 1.
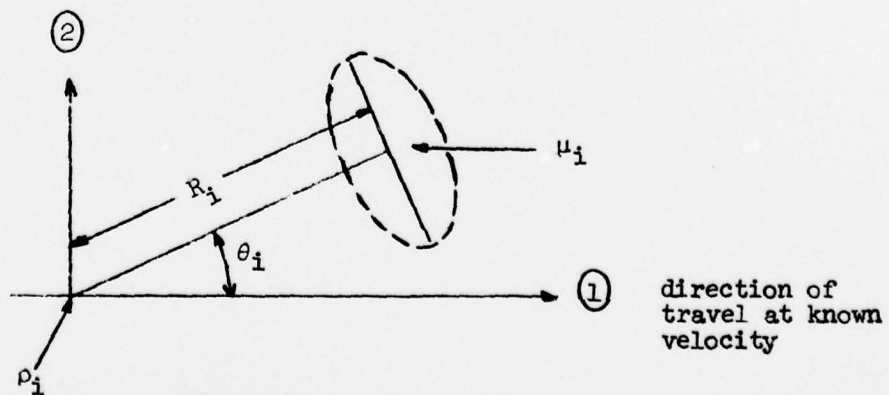

Figure 1

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From elementary geometrical arguments, as in Saunders and Johnson (1964), which need not be given here, we have

$$(5.1) \quad C_i^{-1} = \begin{pmatrix} \sigma_{Ri}^2 \cos^2\theta_i + \sigma_{Ai}^2 \sin^2\theta_i & \dfrac{\sigma_{Ri}^2 - \sigma_{Ai}^2}{2}\sin 2\theta_i \\[2mm] \dfrac{\sigma_{Ri}^2 - \sigma_{Ai}^2}{2}\sin 2\theta_i & \sigma_{Ri}^2 \sin^2\theta_i + \sigma_{Ai}^2 \cos^2\theta_i \end{pmatrix},$$

where at time t_i, σ_{Ri} is the standard deviation of the range error, σ_{Ai} is the standard deviation of the azimuth error and both are known functions of the range Ri, all in accord with Assumption 6°.

We also assume that the matrix D is diagonal and known.

If it is true that the distribution of radar errors is constant in time, that neither the direction of travel of the observatory platform nor its position relative to the target will influence the covariance matrix of observations, then $C_i \equiv C$ for $i = 1,\ldots,n$ and $\Lambda_{ij} = \Lambda_{ji}$. By a straightforward calculation, with the time chosen so that $\bar{t} = \Sigma t_i/n = 0$, we obtain

$$\Sigma_{11} = nC - n^2 C(D+nC)^{-1}C \quad, \quad \Sigma_{12} = \Sigma_{21} = 0 \quad, \quad \Sigma_{22} = n\,\overline{t^2}\,C$$

$$\psi_1 = \bar{z}\Sigma_{11} \quad, \quad \psi_2 = n\,\overline{tz}\,C\,,$$

(where $\bar{z} = \dfrac{1}{n}\Sigma z_i$, $\overline{tz} = \dfrac{1}{n}\Sigma t_i z_i$, $\overline{t^2} = \Sigma t_i^2/n$) and hence

$$S_{11} = \dfrac{C^{-1}}{n} + D^{-1} \quad, \quad S_{12} = S_{21} = 0 \quad, \quad S_{22} = \dfrac{C^{-1}}{n\,\overline{t^2}}\,.$$

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From (4.9), the maximum likelihood estimates are given by

$$\hat{\gamma} = \bar{z} \, , \quad \hat{\kappa} = \overline{tz} / \overline{t^2} \, ,$$

and hence by Theorem 3 and Corollary 2, it follows that $\hat{\mu}_t = \hat{\alpha} + \hat{\beta} t,$ where

$$\hat{\alpha} = \bar{z} - \varepsilon t_0 \, , \quad \hat{\beta} = \frac{\overline{tz}}{\overline{t^2}} + \varepsilon \, .$$

The estimator $\hat{\mu}_t$ is unbiased and has covariance matrix

$$B_t = S_{11} + t^2 S_{22} = \frac{1}{n} \left(1 + \frac{t^2}{\overline{t^2}} \right) C^{-1} + D^{-1} \, .$$

In the case that C is diagonal, the matrix B_t is also diagonal.

In the circumstance that the observations are symmetrically spaced in time and the origin is chosen so that

$$t_i + t_{n+1-i} = 0 \, , \quad \theta_i + \theta_{n+1-i} = \pi \, , \quad i = 1,\ldots,n \, ,$$

it follows by (5.1) that $C_i + C_{n+1-i}$ is a diagonal matrix; the matrices $D + \sum\limits_1^n C_i, \ \Sigma_{11}, \ \Sigma_{22},$ are also diagonal, whereas Σ_{21} is contradiagonal. This facilitates the computation of $S'U'\Lambda US$ which will not in general reduce to S as in Corollary 1.

6. ESTIMATION AND CONFIDENCE INTERVALS FOR SPEED AND HEADING

The parameters of interest when tracking a moving object are the speed and heading, and in this section we consider the accuracy with which they can be estimated. The estimate of the true position μ_t at any time $t,$

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$$\hat{\mu}(t) = \hat{\alpha} + \hat{\beta}t = (\hat{m}_1(t), \hat{m}_2(t)) \quad ,$$

which we write in coordinate form, assuming it follows a linear path. The true velocity s, heading angle h, are given by

$$s = \left\{ [m_1'(t)]^2 + [m_2'(t)]^2 \right\}^{1/2} , \quad h = \arctan\left(\frac{m_2(1) - m_2(0)}{m_1(1) - m_1(0)} \right)$$

These relationships are seen to be simply: if $\beta = (b_1, b_2)$, then

$$(7.1) \qquad\qquad b_1 = s \cos h \quad , \quad b_2 = s \sin h .$$

Thus by analogy we have the equations

$$(7.2) \qquad\qquad \hat{b}_1 = V \cos \Phi \quad , \quad \hat{b}_2 = V \sin \Phi$$

defining the random variables V (for velocity) and Φ for the heading angle, which are estimates of the true speed s and the true heading h.

By Theorem 3, it follows by known results on the marginal distribution of normal variates that $\hat{\beta} \sim \mathcal{N}(\beta, Q_{22})$. The joint density of Φ, V is found by simply making a transformation to polar coordinates. This yields

$$(7.3) \qquad g(\varphi, v | h, s) = \frac{v}{2\pi |Q_i|^{1/2}} \exp\left\{ -\frac{1}{2} \zeta Q_{22}^{-1} \zeta' \right\} , \qquad \begin{array}{c} 0 < v < \infty \\[4pt] 0 < \varphi < 2\pi \end{array}$$

where $\zeta = (v \cos \varphi - s \cos h, v \sin \varphi - s \sin h)$. This density can be used to study the distribution of velocity and heading estimates that could arise under infrequent headings and/or high velocity.

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One might desire separate confidence intervals on the heading
and on the velocity. However, if we proceed to find the marginal
densities of v and φ from (7.3) we see that each density has both
parameters h and s.

Thus a confidence interval for the velocity can be constructed
only if we know the true heading h. Likewise a confidence interval
can be found for the true heading if we know the true velocity. The
presence of the nuisance parameters prevents us from obtaining confi-
dence intervals separately when both parameters are unknown.

However, we can obtain a joint confidence region for (h,s), which
is somewhat inefficient, as follows; from well-known results on
the chi-square distribution of the quadratic form of normal variates
we have

$$(7.4) \qquad P\{(\hat{\beta}-\beta)Q_{22}^{-1}(\hat{\beta}-\beta)' \leq \chi_2^2(p)\} = 1-p ,$$

where $\chi_2^2(p)$ can be easily calculated for any $0 < p < 1$.

The (random) elliptical region W, so defined, determines a
100(1-p) percent confidence region for β. We seek the smallest area
in polar coordinates which is the Cartesian product of intervals and
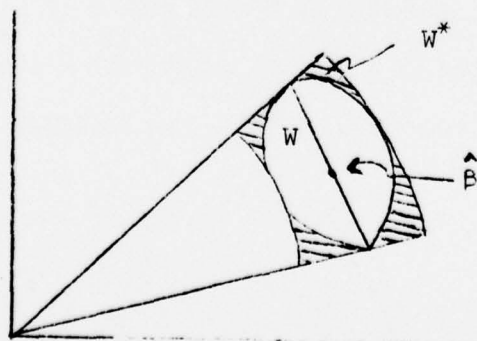contains W; call it W^*. See Figure 2.

Figure 2

Joint confidence intervals for heading angle φ
and velocity V.

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With the obvious notation for the maximum and minimum of argument and modulus; we write

$$(7.5) \qquad W^* = \{(\theta, r): \phi_1 < \theta < \phi_2, \ V_1 < r < V_2\} \ .$$

Now $(r \cos \theta, r \sin \theta) \ \varepsilon \ W$ implies $(\theta, r) \ \varepsilon \ W^*$, $P\{(h,s) \ \varepsilon \ W^*\} \geq 1-p$.

The task to which we now address ourselves is the determination of a functional representation for the random variables (ϕ_i, V_i), $i = 1, 2$. If we denote the elements of the symmetric matrix

$$\frac{1}{\chi_2^2(p)} Q_{22}^{-1} = \begin{pmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{pmatrix} \ ,$$

then we can write from (7.4)

$$(7.6) \qquad q_{11}(x-\hat{b}_1)^2 + 2 \, q_{12}(x-\hat{b}_1)(y-\hat{b}_2) + q_{22}(y-\hat{b}_2)^2 = 1 \ .$$

We want to find the maximum and minimum of the functions

$$(7.7) \qquad f(x,y) = y/x \ , \qquad g(x,y) = x^2 + y^2$$

subject to the restriction (7.6).

Proceeding by analytic geometry or by the method of Lagrange multipliers, leads to the solution of a quartic equation. A simpler method seems to be the following: transform (7.6) to the variables (u,v) via

$$x = \hat{b}_1 + u \cos \theta_0 - v \sin \theta_0 \ ,$$

$$y = \hat{b}_2 + u \sin \theta_0 + v \cos \theta_0 \ ,$$

where

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$$\tan 2\theta_0 = \frac{2q_{12}}{q_{22}-q_{11}} \ .$$

Thus we want to extremize the functions of (u,v) obtained by substituting into (7.7) subject to the restriction $\delta_1 u^2 + \delta_2 v^2 = 1$. Elementary analysis shows that

$$\delta_1 = q_{11} \cos^2\theta_0 + q_{12} \sin 2\theta_0 + q_{22} \sin^2\theta_0 \ ,$$

$$\delta_2 = q_{11} \sin^2\theta_0 - q_{12} \sin 2\theta_0 + q_{22} \cos^2\theta_0 \ .$$

Now let

$$\cos \varphi = \sqrt{\delta_1} \, u \ , \quad \sin \varphi = \sqrt{\delta_2} \, v \ ,$$

and set

$$\eta_i = \frac{\cos \theta_0}{\sqrt{\delta_i}} \ , \quad \xi_i = \frac{\sin \theta_0}{\sqrt{\delta_i}} \ , \quad i = 1,2 \ .$$

The problem then becomes that of extremizing the functions

$$f(\varphi) = \frac{\xi_1 \cos \varphi + \eta_2 \sin \varphi + \hat{b}_2}{\eta_1 \cos \varphi - \xi_2 \sin \varphi + \hat{b}_1}$$

and

$$g(\varphi) = (\eta_1 \cos \varphi - \xi_2 \sin \varphi + \hat{b}_1)^2 + (\xi_1 \cos \varphi + \eta_2 \sin \varphi + \hat{b}_2)^2$$

for the range of values $0 < \varphi \le 2\pi$. It is straightforward to show that $f'(\varphi) = 0$ if and only if

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(7.8)
$$q_{11} + T_1 \sin \varphi + T_2 \cos \varphi = 0 \; ,$$

where

$$q_{11} = \xi_2 \xi_1 + \eta_1 \eta_2 \; , \quad T_1 = \hat{b}_2 \eta_1 - \hat{b}_1 \xi_1 \; , \quad T_2 = \hat{b}_1 \eta_2 + \hat{b}_2 \xi_2$$

with capital letters denoting those random variables which are functions
of the estimates.

Now let Φ_0 be the unique random angle for which $\sin \Phi_0 = T_1/T$,
$\cos \Phi_0 = T_2/T$, where $T = \sqrt{T_1^2 + T_2^2}$. Then (7.8) becomes simply, by
using the trigonometric identity for cosine of a sum, $\cos(\Phi_0 - \varphi) = -\delta_1/T$.

Letting Θ be the angle on $(0, \pi)$ such that $\Theta = \arccos(-\delta_1/T)$
we have, since cosine is an even function, $\Omega_1 = \Phi_0 - \Theta$, $\Omega_2 = \Phi_0 + \Theta$ as
the two solutions of (7.8). Thus we have as the limits on the true
bearing angle as in (7.5), the angles

$$\Phi_1 = \arctan f(\Omega_1) \; , \quad \Phi_2 = \arctan f(\Omega_2) \; .$$

This accomplishes the location of local extremum of f. Graphical examination
of the function on $(0, 2\pi)$ may be necessary to see if these are the true
extrema.

The extremum of g is obtained in a manner similar to that for f.
This yields $g'(\varphi) = 0$ if and only if

(7.9)
$$\delta_1 \sin 2\varphi + \delta_2 \cos 2\varphi + L_1 \sin \varphi + L_2 \cos \varphi = 0$$

where

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$$2\delta_1 = \eta_1^2 - \eta_2^2 + \xi_1^2 - \xi_2^2, \quad \delta_2 = \eta_1\xi_2 - \xi_1\eta_2 ,$$

$$L_1 = \hat{b}_1\eta_1 + \hat{b}_2\xi_1 , \qquad L_2 = \hat{b}_1\xi_2 - \hat{b}_2\eta_2 ,$$

and again upper case letters denote random variables. The solutions of (7.9), call them Γ_1, Γ_2, cannot in general be found explicitly, but must be found numerically. This is easily carried out.

Having found the solutions Γ_i, $i = 1,2$ the limits on the velocity, which were sought, become

$$V_1 = \min(\sqrt{g(\Gamma_1)} , \sqrt{g(\Gamma_2)}) \qquad V_2 = \max(\sqrt{g(\Gamma_1)} , \sqrt{g(\Gamma_2)}) .$$

A circumstance of practical interest occurs whenever the observations are drawn symetrically on each overflight. In this case $\theta_0 = 0$. $\delta_1 = q_{11}$, $\delta_2 = q_{22}$, $\delta_{12} = 0$,

$$\eta_1 = \frac{1}{\sqrt{q_{11}}} , \quad \xi_2 = 0 = \xi_1 , \quad \eta_2 = \frac{1}{\sqrt{q_{22}}}$$

$$2\delta_1 = \frac{1}{q_{11}} - \frac{1}{q_{22}} , \; \delta_2 = 0 \quad T_1 = b_1/\sqrt{q_{11}} , \; T_2 = -b_2/\sqrt{q_{22}} \quad .$$

Thus the equation (7.9) reduces to

$$(7.10) \qquad \frac{1}{2} \left(\frac{1}{q_{11}} - \frac{1}{q_{22}}\right) \sin 2\varphi + \frac{\hat{b}_1}{\sqrt{q_{11}}} \sin \varphi = \frac{\hat{b}_2}{\sqrt{q_{22}}} \cos \varphi .$$

If we further specialize and assume that $q_{11} = q_{22}$ we see from (7.10) that the equation to be solved is merely $\tan \varphi = \hat{b}_2/\hat{b}_1$, which has solutions

$$\cos \varphi = \pm \frac{\hat{b}_1}{V} , \quad \sin \varphi = \pm \frac{\hat{b}_2}{V} ,$$

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where V is the estimate of velocity defined in (7.2). From the definition of g previously given we see that $q_{11} = q_{22}$ implies that the function we were extremizing was merely

$$g(\varphi) = \left(\frac{\cos \varphi}{\sqrt{q_{11}}} + \hat{b}_1 \right)^2 + \left(\frac{\sin \varphi}{\sqrt{q_{11}}} + \hat{b}_2 \right)^2$$

and we obtain, directly from the definition of V_1 and V_2 previously made, the simple answer

$$V_2 = V + \frac{1}{\sqrt{q_{11}}} \, , \quad V_1 = V - \frac{1}{\sqrt{q_{11}}} \, .$$

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This paper presents a general theorem on the invariant behavior of a certain function of a matrix. It then shows the importance of this result principally by using it to derive properties of certain maximum likelihood estimates which arise when considering problems such as the location of a moving object being surveyed from a moving observatory when all data on location are subject to stochastic error. This problem is important in tracking objects either from an observatory satellite or from a transport plane bearing ground seeking radar. Some applications to this situation are made.